7.4 Derivative, Integral, Multiplication of Laplace Transform

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\cancel{\mathcal{L}\left\{\int_0^\infty f(t)dt\right\}} = \frac{F(s)}{s}$ correction: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$

$F(s-a) = \mathcal{L}\{e^{at} f(t)\}$

let's look at $F'(s)$. what happens in t-domain?

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$F'(s) = \frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty f(t)\left(\frac{d}{ds} e^{-st}\right) dt = \int_0^\infty f(t) \cdot -t e^{-st} dt$$

$$= \int_0^\infty [-t f(t)] e^{-st} dt$$

$$\boxed{F'(s) = \mathcal{L}\{-t f(t)\}}$$

repeat, $F''(s) = \mathcal{L}\{t^2 f(t)\}$ each differentiation is a factor of $-t$

so, $\boxed{F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}}$

extra

this can help us with, for example, $\mathcal{L}\{t \underbrace{\cosh(6t)}_{\text{on the table}}\}$

look at $F'(s) = \mathcal{L}\{-t f(t)\}$

$$\underset{\text{deriv. in s-domain}}{F'(s)} = \mathcal{L}\{t \cdot \underbrace{(-\cosh(6t)}_{f(t) \to F(s) \quad \text{(table)}}\}$$

$$\mathcal{L}\{-\cosh(6t)\} = F(s)$$

$$= -\frac{s}{s^2-6^2}$$

$$\frac{d}{ds}\left(-\frac{s}{s^2-6^2}\right) = \cdots = \boxed{\frac{s^2+36}{(s^2-36)^2} = \mathcal{L}\{t\cosh(6t)\}}$$

if $F(s) = \underbrace{\ln\left(\frac{1}{s^2-16}\right)}_{\text{not on table}}$, what is $f(t)$?

$F(s) = \mathcal{L}\{-t\,f(t)\}$

$$F(s) = \ln(1) - \ln(s^2-16)$$

$$= 0 - \ln[(s+4)(s-4)]$$

$$= -[\ln(s+4) + \ln(s-4)] = -\ln(s+4) - \ln(s-4)$$

$$F'(s) = -\underbrace{\frac{1}{s+4}}_{} - \underbrace{\frac{1}{s-4}}_{}$$

on table

$F'(s) = \mathcal{L}\{-t\,f(t)\}$

$$\frac{\mathcal{L}^{-1}\{F'(s)\}}{-t} = f(t)$$

$$f(t) = \frac{\mathcal{L}^{-1}\left\{-\frac{1}{s+4} - \frac{1}{s-4}\right\}}{-t}$$

$$= \frac{-e^{-4t} - e^{4t}}{-t} = \boxed{\frac{e^{-4t} + e^{4t}}{t}}$$

now integration in s-domain, specifically $\int_s^\infty F(\sigma)\,d\sigma$ dummy variable

start with $F(s) = \int_0^\infty f(t)e^{-st}dt$

then $\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \int_0^\infty f(t)\,e^{-\sigma t}\,dt\,d\sigma$

swap integration order

$$\int_s^\infty F(\sigma)\,d\sigma = \int_0^\infty f(t)\left(\int_s^\infty e^{-\sigma t}\,d\sigma\right)dt$$

$$= \int_0^\infty f(t)\left(-\frac{1}{t}e^{-\sigma t}\Big|_{\sigma=s}^{\sigma=\infty}\right)dt$$

$$= \int_0^\infty f(t)\left(0 + \frac{1}{t}e^{-st}\right)dt$$

$$= \int_0^\infty \left(\frac{f(t)}{t}\right)e^{-st}\,dt$$

$$\boxed{\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left\{\frac{f(t)}{t}\right\}}$$

$\lim_{t\to 0^+} \frac{f(t)}{t}$ must exist useful for Laplace transform of something over t

for example, $\mathcal{L}\left\{\frac{1-\cos(t)}{t}\right\}$ → has Laplace transform on table

$$\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

check if $\lim\limits_{t\to 0^+} \frac{1-\cos(t)}{t}$ exists

$$= \lim_{t\to 0^+} \frac{\sin(t)}{1} = 0$$

$$\mathcal{L}\{1-\cos(t)\} = \frac{1}{s} - \frac{s}{s^2+1}$$ (table lookup)

$$\mathcal{L}\left\{\frac{1-\cos(t)}{t}\right\} = \int_s^\infty \left(\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+1}\right)d\sigma$$

$$= \ln\sigma - \tfrac{1}{2}\ln(\sigma^2+1)\Big|_s^\infty$$

$$= \ln\sigma - \ln\sqrt{\sigma^2+1}\Big|_s^\infty$$

$$= \ln\left(\frac{\sigma}{\sqrt{\sigma^2+1}}\right)\Bigg|_s^\infty = \ln(1) - \ln\left(\frac{s}{\sqrt{s^2+1}}\right)$$

$$= \boxed{-\ln\left(\frac{s}{\sqrt{s^2+1}}\right)}$$

let's revisit $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$= \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$\lim_{s\to 0}\left[sF(s) - f(0)\right] = \lim_{s\to 0}\int_0^\infty f'(t)\, e^{-st}\, dt$$

$$\lim_{s\to 0} sF(s) - \cancel{f(0)} = \int_0^\infty f'(t)\, dt$$

$$= \lim_{t\to\infty} f(t) - \cancel{f(0)}$$

$$\boxed{\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)}$$

Final value theorem

on HW 5: $\int_0^\infty \frac{\sin(t)}{t}\,dt$ using final value theorem

define $g(t) = \int_0^t \frac{\sin(\tau)}{\tau}\,d\tau$, find $\lim_{t\to\infty} g(t)$